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一类奇异边值问题和壁摩擦的估计公式

徐云滨　董媛媛
（榆林学院数学与应用数学系，榆林 719000）

摘　要　利用单调逼近方法给出了源于流体边界层理论中的一类奇异边值问题正解的存在性和唯一性的充分条件，同时给出了用压力梯度参数表示的壁摩擦的估计公式；并且数值结果证明了估计公式的可靠性和有效性。

关键词　奇异边值问题　　存在性和唯一性　　壁摩擦　　估计式
中图法分类号　O175.8；　　　**文献标志码**　A

Falkner-Skan equation is a classic laminar boundary layer equation. It was first deduced by U. M. Falkner and S. W. Skan in 1931[1]. The numerical solutions of Falkner-Skan equation was studied by D. R. Hartree[2]. Later, an numerical method for solving Falkner-Skan equation was also presented by Guojun Li[3], Liancun Zheng Anguo Wen and Xinxin Zhang provided the approximate analytical solutions by using Adomian decomposition method. Many works had been investigated about the Falkner-Skan equation, for details, see the reference [5—7]. All of the above-mentioned works have had attention paid to the analytical solutions or numerical ones. The qualitative properties of the solutions are studied in this paper and a theoretical estimate formula for skin friction coefficient denoted by the pressure gradient parameter is presented.

1　Falkner-Skan equation

$$f'''(\eta)+f(\eta)f''(\eta)+\beta(1-f'^2(\eta))=0, 0\leqslant\eta<+\infty \quad (1)$$

$$f(0)=0, f'(0)=0, f'(+\infty)=1 \quad (2)$$

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第一作者简介：徐云滨(1979—)，男，山东临沂人，讲师，硕士，研究方向：微分方程理论。E-mail：xuyunbin886@sina.com。

Introducing a transformation as refs. [8,9]

$$g(t)=f''(\eta)\ (\text{dimensionless shear stress}) \quad (3)$$

$$t=f'(\eta)\ (\text{dimensionless tangential velocity}) \quad (4)$$

and substitute equaes (3), (4) into equaes (1), (2), in terms of $f''(\eta)>0, 0<\eta<+\infty$, $f''(+\infty)=0$, we arrive at the following singular nonlinear two-point boundary value problems:

$$\begin{cases} g''(t)=-\beta\left(\dfrac{1-t^2}{g(t)}\right)'-\dfrac{t}{g(t)}, & 0<t<1 \\ g(1)=0, & g(0)g'(0)=-\beta \end{cases} \quad (5)$$

It may be seen from the derivation process that only the positive solutions of eq. (5) are physically significant.

2　The solutions of eq. (5)

Since the problem is singular at $t=1$, it is convenient by considering the boundary conditions without singularities

$$\begin{cases} g''(t)=-\beta\left(\dfrac{1-t^2}{g(t)}\right)'-\dfrac{t}{g(t)}, & 0<t<1 \\ g(1)=h, & g(0)g'(0)=-\beta \end{cases} \quad (6)$$

Denote the solution of eq. (6) by $g_h(t)$, we first show the following lemmas.

Lemma 1　If $h_1>h_2>0$, and $\beta\geqslant0$, then $g_{h_1}(t)\geqslant g_{h_2}(t)$.

Proof If the inequality is not true, then there exists a point $t_0 \in [0,1)$ such that $g_{h_1}(t_0) < g_{h_2}(t_0)$. We consider only two cases.

2.1 $g_{h_1}(0) < g_{h_2}(0)$

Choose $t_0 = 0$, since $g_{h_1}(1) > g_{h_2}(1) > 0$, then there exists a maximal interval $[0,k]$ $(k<1)$ such that $g_{h_1}(t) < g_{h_2}(t)$ for $t \in [0,k)$, and $g_{h_1}(k) = g_{h_2}(k) = m > 0$. $g_{h_1}(t)$ and $g_{h_2}(t)$ are both the positive solutions of the integral equation

$$g(t) = m + \int_0^k G(t,s)\frac{1}{g(s)}\mathrm{d}s \tag{7}$$

Where the Green's function $G(t,s)$ is defined as

$$G(t,s) = \begin{cases} s(1-t), & 0 \leqslant s \leqslant t < k \\ (1-s)(\beta + \beta s + s), & 0 \leqslant t \leqslant s < k. \end{cases}$$

Eq. (7) implies:

$$0 < g_{h_2}(t) - g_{h_1}(t) = \int_0^k G(t,s)\left[\frac{1}{g_{h_2}(s)} - \frac{1}{g_{h_1}(s)}\right]\mathrm{d}s < 0.$$

which is a contradiction.

2.2 $g_{h_1}(0) \geqslant g_{h_2}(0)$

Since $g_{h_1}(1) > g_{h_2}(1) > 0$, then there exists a maximal interval $[a,b]$ $(0 \leqslant a < b < 1)$, which contains the point t_0 such that $g_{h_1}(a) = g_{h_2}(a)$ and $g_{h_1}(b) = g_{h_2}(b)$, and $g_{h_1}(t) < g_{h_2}(t)$ for $t \in (a,b)$. Let $g(t) = g_{h_2}(t) - g_{h_1}(t)$ then $g(t)$ has a positive maximum at $t_0 \in (a,b)$ and $g'(t_0) = 0$. Denote $g(t_0) = m$, for $g''(t) = -\beta\left(\frac{1-t^2}{g(t)}\right)' - \frac{t}{g(t)}$, intergation from t_0 to s yields $g'(s) = -\beta\frac{1-s^2}{g(s)} - \int_{t_0}^s \frac{t}{g(t)}\mathrm{d}t + \beta\frac{1-t_0^2}{m}$, and again integration from t_0 to b leads to

$$g(b) - m = -\beta\int_{t_0}^b \frac{1-s^2}{g(s)}\mathrm{d}s - \int_{t_0}^b \frac{t(b-t)}{g(t)}\mathrm{d}t + \beta\frac{(1-t_0^2)(b-t_0)}{m},$$

$$g_{h_1}(b) - m = -\beta\int_{t_0}^b \frac{1-s^2}{g_{h_1}(s)}\mathrm{d}s - \int_{t_0}^b \frac{t(b-t)}{g_{h_1}(t)}\mathrm{d}t + \beta\frac{(1-t_0^2)(b-t_0)}{m} \tag{a}$$

$$g_{h_2}(b) - m = -\beta\int_{t_0}^b \frac{1-s^2}{g_{h_2}(s)}\mathrm{d}s - \int_{t_0}^b \frac{t(b-t)}{g_{h_2}(t)}\mathrm{d}t + \beta\frac{(1-t_0^2)(b-t_0)}{m} \tag{b}$$

equaes (a), (b) leads to

$$0 = g_{h_1}(b) - g_{h_2}(b) = \beta\int_{t_0}^b (1-s^2)\left(\frac{1}{g_{h_2}(s)} - \frac{1}{g_{h_1}(s)}\right)\mathrm{d}s + \int_{t_0}^b t(b-t)\left(\frac{1}{g_{h_2}(t)} - \frac{1}{g_{h_1}(t)}\right)\mathrm{d}t < 0.$$

Which is also a contradiction.

Lemma 2 For any fixed $h > 0$ and $\beta \geqslant 0$, eq. (6) has at most one positive solution.

Proof Suppose eq. (6) has two positive solutions $g_1(t)$ and $g_2(t)$ for each fixed $h > 0$ and $\beta \geqslant 0$. Then, without loss of generality, we may assume that there exists a point $t_0 \in [0,1]$ such that $g_1(t_0) > g_2(t_0)$. Since $g_1(1) = g_2(1) = h$ then there exists a maximal close interval $[a_1, b_1] \subseteq [0,1]$ such that $g_1(t) > g_2(t)$ for
$t \in [a_1, b_1]$.

(i) If $a_1 = 0$, then $g_1(t) \geqslant g_2(t)$ for
$t \in [0, b_1] \subseteq [0,1]$ and $g_1(b_1) = g_2(b_1)$.

(ii) If $a_1 \neq 0$, then $g_1(a_1) = g_2(a_1)$ 和 $g_1(b_1) = g_2(b_1)$ for $t \in [a_1, b_1] \subset [0,1]$, and $g_1(t) > g_2(t)$. for $t \in (a_1, b_1)$.

It follows along the same lines as the cases (i) and (ii) in lemma 1, we may show that this is impossible.

Lemma 3 For any fixed $h > 0$ and $\beta \geqslant 0$, eq. (6) have at least one positive solution $g_h(t)$.

Proof For any fixed $h > 0$ and $\beta \geqslant 0$, if $g(t)$ is the positive solution of eq. (6), then $g(t)$ must be a positive solution of the integral equation

$$g(t) = h + \int_0^1 G(t,s)\frac{1}{g(s)}\mathrm{d}s \tag{8}$$

Where the Green's function $G(t,s)$ is defined as

$$G(t,s) = \begin{cases} s(1-t), & 0 \leqslant s \leqslant t < 1 \\ (1-s)(\beta + \beta s + s), & 0 \leqslant t \leqslant s < 1. \end{cases}$$

We defining a mapping T:

$$Tg(t) = h + \int_0^1 G(t,s)\frac{1}{g(s)}\mathrm{d}s.$$

Where $\Omega=\{g(t)\in C[0,1]:h\leqslant g(t)\leqslant(Th)(t)\}$, and $C[0,1]$ is the set of all real-valued continuous functions defined on $[0,1]$. Then T is a completely continuous mapping from Ω to Ω. The Schauder Fixed Point Theorem[9] asserts that the mapping Thas at least one fixed point$g_h(t)$ in Ω, which implies that $g_h(t)$ is a positive solution of eq. (6).

Denote $g(0)=\sigma$ and consider the initial value problem

$$\begin{cases} g''(t)=-\beta\left(\dfrac{1-t^2}{g(t)}\right)'-\dfrac{t}{g(t)},0<t<1 \\ g(0)=\sigma, \qquad g'(0)=-\dfrac{\beta}{\sigma} \end{cases} \tag{9}$$

Let $g(t)$ be the positive solution of eq. (5) and $[0,t_\sigma^*)$ be the maximal interval of existence with $g(0;\beta)=\sigma$, then we may established the following results:

Lemma 4 (i) Let g_1 and g_2 be solutions for $\sigma=\sigma_1$ and $\sigma=\sigma_2$, if $\sigma_1<\sigma_2$, then $t_{\sigma_1}^*<t_{\sigma_2}^*$.

(ii) t_σ^* is a continuous function of σ and $t_\sigma^*\to+\infty$ as $\sigma\to+\infty$.

The proof of this lemma is similar to that of lemmas 1, 2 and 3 in ref. [12], we omitted here.

Lemma 5 For any fixed $h>0$ and $\beta\geqslant0$, the positive solutions $g_h(t)$ of eq. (6) satisfies

$$g_h(0,\beta)>\sqrt{\frac{1+4\beta}{6}}.$$

Proof In terms of eq. (9), for $t\in(0,1)$

$$g(t)<\sigma+\frac{\beta}{3\sigma}-\frac{\beta t}{\sigma}-\frac{t^3}{6\sigma}.$$

Let $f(t)=\sigma+\dfrac{\beta}{3\sigma}-\dfrac{\beta t}{\sigma}-\dfrac{t^3}{6\sigma}$, then the positive solution of initial value equae (9) satisfies $g(t)<f(t)$ for $t\in(0,1)$. In term of lemma 4 assume $f(t)$ intersects the t-axis at the point t_0^*. Especially for $t_\sigma^*=1$, this yields $\sigma=\sqrt{\dfrac{1+4\beta}{6}}$ for $\beta\geqslant0$.

It is similar to lemma 1, we may show the positive solution of eq. (9) is increasing with σ, so the positive solutions $g(t;\sigma)$ of eq. (9) can't intersect the point 1 for $\sigma\leqslant\sqrt{\dfrac{1+4\beta}{6}}$. It implies that for $\sigma\leqslant\sqrt{\dfrac{1+4\beta}{6}}$, the positive solutions of initial value problem equae (9) satisfies $g(1)<0$. This shows that for any fixed $h>0$ and $\beta\geqslant0$, the positive solutions $g_h(t)$ of eq. (6) satisfies

$$g_h(0,\beta)>\sqrt{\frac{1+4\beta}{6}}.$$

Theorem Assume that $\beta\geqslant0$, then eq. (5) has a unique positive solution.

Proof lemma 2 and lemma 3 show that for any $h>0$and $\beta\geqslant0$ eq. (6) has a unique positive solution. Then for any $h_2>h_1>0$, in terms of eq. (8) and lemma 1,

$$0<g_{h_2}(t)-g_{h_1}(t)=h_2-h_1+\int_0^1 G(t,s)\left[\frac{1}{g_{h_2}(s)}-\frac{1}{g_{h_1}(s)}\right]ds\leqslant h_2-h_1.$$

It indicates the series of positive solutions $\{g_h(t)\}$ converges to a limit uniformly with h on $[0,1]$, denotes by $g_0(t)$. Then $\lim\limits_{h\to0}g_h(t)=g_0(t),t\in[0,1]$.

Lemma 5 implies $g_0(0,\beta)>\sqrt{\dfrac{1+4\beta}{6}}(\beta\geqslant0)$.

For any $h\geqslant0$, by the convexity of $g_h(t)$, this yields

$$\begin{aligned} g_h(t)&\geqslant h+(g_h(0,\beta)-h)(1-t)\geqslant \\ &h+g_h(0,\beta)(1-t)-h= \\ &g_h(0,\beta)(1-t)\geqslant\sqrt{\frac{1+4\beta}{6}}(1-t) \end{aligned} \tag{10}$$

In terms of eq. (8) and using the above, we can see that

$$\begin{aligned} g_h(t)&=h+\int_t^1\frac{(1-s)(\beta+\beta s+s)}{g(s)}ds+\int_0^t\frac{s(1-t)}{g(s)}ds\leqslant \\ &h+\sqrt{\frac{6}{1+4\beta}}\left(\frac{1+3\beta}{2}-\beta t-\frac{\beta}{2}t^2\right) \end{aligned} \tag{11}$$

So $g_h(0,\beta)\leqslant h+\dfrac{\sqrt{6}(1+3\beta)}{2\sqrt{1+4\beta}}$, by using the Monotone Convergence Theorem[13—15], Letting $h\to0^+$ in the integral equation (8), we get

$$g_0(t)=\int_0^1 G(t,s)\frac{1}{g_0(s)}ds.$$

The above arguments indicate that eq. (5) has a unique positive solution $g_0(t)$. Furthermore, in terms of lemma 5, we have

$$\sqrt{\frac{1+4\beta}{6}} \leqslant \sigma = g_0(0,\beta) \leqslant \frac{\sqrt{6}(1+3\beta)}{2\sqrt{1+4\beta}}.$$

This proves that eq. (5) have a unique positive solution $g(t)$, satisfying

$$\sqrt{\frac{1+4\beta}{6}} \leqslant \sigma = g_0(0,\beta) \leqslant \frac{\sqrt{6}(1+3\beta)}{2\sqrt{1+4\beta}} \quad (12)$$

In order to illustrate the reliability and efficiency of the proposed theoretical results and the estimate formula. Denote the skin friction coefficient $g(0,\beta)$ obtained numerically by $\sigma_{com} = g(0,\beta)$, and the estimated results obtained by estimation formula (12) by $\sigma_{\text{lower-bound}} = \sqrt{\frac{1+4\beta}{6}}$ and $\sigma_{\text{upper-bound}} = \frac{\sqrt{6}(1+3\beta)}{2\sqrt{1+4\beta}}$, respectively.

A comparison is presented in table 1. The reliability and efficiency of the theoretical prediction are verified by numerical results.

Table 1 Comparison numerical results

β	$\sigma_{\text{lower-bound}}$	$\sigma_{\text{com}} = g(0,\beta)$	$\sigma_{\text{upper-bound}}$
$\beta=0$	0.408 2	0.457 7	1.224 7
$\beta=0.05$	0.447 2	0.531 1	1.285 7
$\beta=0.1$	0.483 0	0.587 0	1.345 6
$\beta=0.2$	0.547 7	0.686 7	1.460 6
$\beta=0.3$	0.605 5	0.774 8	1.568 9
$\beta=0.4$	0.658 3	0.854 4	1.671 0

Table 1 shows all numerical results lie in the range that are estimated by formula (12). When pressure gradient parameter is bigger, the error that is estimated by lower-bound of formula (12) is bigger. But, with the decreasing of gradient parameter, the formula (12) goes more and more reliable and efficient. Especially, for appropriately small pressure gradient parameter, we can consider the results that are estimated by lower-bound of formula (12) as approximate value of skin friction coefficient. Clearly, all results are completely consistent with the results obtained by theoretical analysis.

4 Conclusions

This paper presents a theoretical analysis for the boundary layer flow. The boundary layer equations are reduced into a singular nonlinear two-point boundary value of ordinary differential equation when Crocco variables were introduced. Sufficient conditions for existence and uniqueness of positive solutions are established. Furthermore, a theoretical estimate formula for the skin friction coefficient is given. The reliability and efficiency of the theoretical prediction are verified by numerical results.

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A Class of Singgular Boundary Value Problem and a Theoretical Estimate Formula for Skin Friction Coefficient

XU Yun-bin, DONG Yuan-yuan

(Department of Mathematics and Applied Mathematics, Yulin University, Yulin 719000, P. R. China)

[**Abstract**] A class of singular nonlinear boundary value problems arising in boundary layer theory are studied. Sufficient conditions for the existence and uniqueness of the positive solutions to the problem are established by utilizing the monotonic approaching technique. And at theoretical estimate formula for skin friction coefficient is presented. The formula can be successfully applied to estimate the value of the skin friction coecient. The correctness of the analytical predictions is verified by the numerical results.

[**Key words**] singular boundary value problem existence and uniqueness skin friction estimation

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Matroid Unions and Transversal Matroids

LÜ Guo-liang, YU Bao-min

(Department of Mathematics and Information Science, Weinan Teachers University, Weinan 714000, P. R. China)

[**Abstract**] The matroid unions and transversal matroids are investigated. Firstly, rank function of the matroid unions defined on varied set is deduced. Then, combining matroid determined by polymatroid function with rank function of transversal matiroid, several relations and properties of matroid unions and transversal matroids are given.

[**Key words**] matroid unions transversal matroid polymatroid function rank maximum common independent set problem