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论 文

数学

一类二阶非线性差分方程多重周期解的存在性

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摘 要 用 Z_2 群指标理论探讨了一类二阶非线性差分方程多重周期解的存在性,得到了该类差分方程多重周期解存在的充分条件,并给出了详细证明。最后,用一个例子说明了结果的合理性。
关键词 周期解 非线性差分方程 Z_2-群指标理论
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Attention has been paid to the existence and multiplicity of periodic solutions for second-order nonlinear differential equations [1—3]. In this paper, we will investigate the difference equations corresponding to the following differential equation:

$$x''(t-\tau)+\sigma f(t,x(t),x(t-\tau),x(t-2\tau))=x(t-\tau) \quad (1)$$

where σ is a constant. In ref. [3], the authors investigated the qualitative properties of an analogue of equa. (1). The sufficient conditions responsible for the existence of multiple periodic solutions of equa. (1) were obtained by a variational method (see ref. [3]). However, the contributions of ref. [3] are referred to the continuous differential-equation models. In applied sciences, the models are generally described as discrete ones. Especially, in earth sciences the difference equations is a dominant form that describes atmospheric and oceanic motions. It is therefore obvious that the exploration of the qualitative properties for nonlinear difference equations is urgently needed refs. [4—7]. Upon this requirement, we will use the Z_2-group index theory to explore the periodic solutions of the discrete version of equa. (1). Since the difference equations are discrete and then finitely dimensional, the traditional ways of establishing the functional in ref. [3] may be inapplicable. Then an appropriate variational structure will be developed to address the periodic solutions of the discrete equation.

Denote by N, Z, R the set of the natural numbers, integers, and real numbers, respectively, the discrete version of equa. (1) can be written as follows by the frog-leap finite difference scheme

$$\left(\frac{\partial^2 x}{\partial t^2}\right)_n=\frac{x_{n+1}-2x_n+x_{n-1}}{(\Delta t)^2}=\frac{\Delta^2(x_n)}{(\Delta t)^2}:$$

$$\Delta^2(x_n)+\tau^2\sigma f(n+1,x_{n+1},x_n,x_{n-1})=\tau^2 x_n \quad (2)$$

where the time step Δt is assumed to be τ. Without loss of the generality, we discuss the periodic solutions of

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the following difference equations

$$\Delta^2(x_n) - x_n + \sigma f(n, x_{n+1}, x_n, x_{n-1}) = 0 \qquad (3)$$

Where

(A_1) $f: Z \times R^3 \to R$ is a continuous functional in the second, the third and the fourth variables and satisfies $f(t+m, u, v, w) = f(t, u, v, w)$ for a given positive integer m;

(A_2) there exist a continuously differentiable function $F: Z \times R^2 \to R$, such that

$F'_v(t, u, v) + F'_u(t-1, v, w) = f(t, u, u, w)$;

(A_3) F satisfies: $F(t, -u, -v) = F(t, u, v)$, and $f(t, -u, -v, -w) = -f(t, u, v, w)$.

1 Variational structure of equa. (3)

To discuss the existence of periodic solutions for equa. (3), we construct a variational problem and then reduce the periodic solutions for equa. (3) into the critical points of the corresponding functional.

Let $X = \{x = \{x_n\}: x_n \in R, n \in Z\}$ be a set of sequences. For any $x, y \in X$, $a, b \in R$, $ax + by$ is defined as

$$ax + by = \{ax_n + by_n\} \in X \qquad (4)$$

Then X is vector space. To facilitate the discussion, we define a sub-space of X as follows.

$$E_{pm} = \{x = \{x_n\} \in X \mid x_{n+pm} = x_n, n \in Z\} \qquad (5)$$

where p and m are two given positive integers. It is obvious that E_{pm} is isomorphic to R^{pm}. If we define a inner product on E_{pm} by

$$<x, y> E_{pm} = \sum_{i=1}^{pm} x_i y_i, \forall x \in E_{pm} \qquad (6)$$

then the norm $\|\cdot\|$ can be induced by

$$\|x\| = \left(\sum_{j=1}^{pm} x_j^2\right)^{\frac{1}{2}}, \forall x \in E_{pm} \qquad (7)$$

Thus, E_{pm} with the inner product in equa. (6) is a finite dimensional Hilbert space and linearly homeomorphic to R^{pm}.

A functional on E_{pm} can be constructed as follows.

$$J(x) = \sum_{n=1}^{pm} \left[\frac{1}{2}(|x_{n+1} - x_n|^2 + |x_n|^2) - \sigma F(n, x_{n+1}, x_n)\right] \qquad (8)$$

where F is the one in the assumption (A_2) as mentioned above.

It is easily shown that $J \in C^1(E_{pm}, R)$. For any $x = \{x_n\} \in E_{pm}$, the first-order variational of J can be derived as follows.

$$\frac{\partial J}{\partial x_n} = -\{\Delta^2(x_n) - x_n + \sigma f(n, x_{n+1}, x_n, x_{n-1})\},$$
$$n = 1, 2, 3 \cdots, pm \qquad (9)$$

Therefore, equa. (8) is the variational problem of equa. (3). We will use Z_2-group index theory to explore the critical points of equa. (8). In next section, some definitions and basic lemmas are stated.

2 Definitions and lemmas

Let S be a real Hilbert space, and I be a continuously Fréchet differentiable functional, *i. e.* $I \in C^1(S, R)$.

Definition 2.1 A "critical point" of the functional I is a point $x \in S$ for which $I'(x) = 0$. A "critical value" of I is a number c such that $I(x) = c$ for some critical point x. The set $K = \{x \in S \mid I'(x) = 0\}$ is the "critical set" of I. We denote by K_c the set $\{x \in S \mid I'(x) = 0, I(x) = c\}$. The "critical level" set I_c of I is defined by $I_c = \{x \in S \mid I(x) \leqslant c\}$.

Definition 2.2 I is said to satisfy P-S condition, when a sequence $\{s_n\} \subset S$ is of the property: $\{I(s_n)\}$ is bounded and $I'(s_n) \to 0 (n \to \infty)$, then $\{s_n\}$ possesses a convergent subsequence in S.

A closed symmetric set $A \subset S$ is said to satisfy property ζ if there exists an odd continuous function ϕ: $A \to R^n \setminus \{\theta\}$ for some $n \in Z^+$. Let $N_A \subset Z$ be defined as follows: $n \in N_A$ if and only if A satisfies proprty ζ with this n.

Definition 2.3 Let $\sum = \{A \mid A \subset S \setminus \{\theta\}$, and A is closed, symmetric set}. Define $\gamma:\Sigma\to Z^+\cup\{+\infty\}$ as follows:

$$\gamma(A)=\begin{cases}\min N_A, & \text{if } N_A\neq\emptyset,\\ 0, & \text{if } A=\emptyset,\\ +\infty, & \text{if } A\neq\emptyset, \text{but } N_A=\emptyset\end{cases}\tag{10}$$

Then γ is the genus of Σ. We let $i_1(I)=\lim\limits_{a\to -0}\gamma(I_a)$ and $i_2(I)=\lim\limits_{a\to-\infty}\gamma(I_a)$.

Assume that B_δ is an open ball in S with radius δ and center 0, we introduce the Lemma 3.1 related to Z_2-group index theory.

Lemma 2.1[8]　Let $I\in C^1(S,R)$ be an even functional that satisfies the P-S condition and $I(\theta)=0$.

(B_1) If there exists an m-dimensional subspace S_1 of S and $\delta>0$, such that

$$\sup_{x\in S_1\cap B_\delta} I(x)<0,$$

then $i_1(I)\geqslant m$;

(B_2) If there exists a j-dimensional subspace $\tilde{S}$ of S such that

$$\inf_{x\in\tilde{S}^\perp} I(x)>-\infty,$$

then $i_2(I)\leqslant j$.

If $m\geqslant j$, and the conditions (B_1) and (B_2) hold, I has at least $2(m-j)$ different critical points.

3　Main results

Theorem 3.1　If the difference equations equa. (3) satisfy the following conditions:

(C_1) $F(t,0,0)=0$, and $\dfrac{\partial f(t,u,v,w)}{\partial t}\neq 0$, $\forall (t,u,v,w)\in R^4$,

(C_2) there exists a constant $\beta>0$ such that $F(t,u,v)<0$ whenever $u^2+v^2>\beta, t\in R$.

(C_3) $\lim\limits_{|\rho|\to 0}\dfrac{F(t,u,v)}{|\rho|^2}=1$, $|\rho|=\sqrt{u^2+v^2}$,

and for $\sigma>\dfrac{9}{4}$ and a given positive integer p, the equa. (3) has $2pm$ nontrivial pm-periodic solutions.

Lemma 3.1　Assume that the functions $f(t;u,v,w)$ and $F(t,u,v)$ satisfy the conditions (C_1) and (C_2) of Theorem 3.1, the functional J in section 1 is bounded for all $x\in E_{pm}$, and then satisfies the P-S condition.

Proof For $x\in E_{pm}$, we have

$$J(x)=\sum_{n=1}^{pm}\left[\frac{1}{2}|x_{n+1}-x_n|^2+\frac{1}{2}|x_n|^2-\sigma F(n,x_{n+1},x_n)\right]=$$

$$\sum_{n=1}^{pm}\left[x_n^2-x_nx_{n+1}+\frac{1}{2}x_n^2-\sigma F(n,x_{n+1},x_n)\right]=$$

$$\sum_{n=1}^{pm}\left[\frac{3}{2}x_n^2-x_nx_{n+1}-\sigma F(n,x_{n+1},x_n)\right]=$$

$$x^{\mathrm T}Ax-\sigma\sum_{n=1}^{pm}F(n,x_{n+1},x_n),$$

where $x=(x_1,x_2,\cdots,x_{pm})^{\mathrm T}$,

$$A=\begin{pmatrix}3/2 & -1/2 & 0 & \cdots & 0 & -1/2\\ -1/2 & 3/2 & -1/2 & \cdots & 0 & 0\\ 0 & -1/2 & 3/2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -1/2 & 0 & 0 & \cdots & -1/2 & 3/2\end{pmatrix}_{pm\times pm}.$$

The matrix has pm eigenvalues. Let $\lambda_1,\lambda_2,\lambda_3,\cdots,\lambda_{pm}$ be the eigenvalues of A. By matrix theory, it is shown that $\lambda_j>0, j=1,2,3,\cdots,pm$.

Set　$\lambda_{\min}=\min\limits_{j=1,2,\cdots,pm}\lambda_j$, $\lambda_{\max}=\max\limits_{j=1,2,\cdots,pm}\lambda_j$.

Let $X=\{(u,v)\,|\,u^2+v^2>\beta\}$. Then X is open and $X\subset R^2$. It is easily known $X^\perp$ is closed in R^2. From the condition (C_2) of Theorem 3.1, we obtain that $F(t,u,v)$ has an upper bound for $(u,v)\in X$ and $t\in R$. On the other hand, F is a continuously differentiable function on R^3, and then F is bounded on the closed set $X^\perp$ for a finite time t. It is therefore derived that F is bounded from above on R^2 for a finite time t. That is, there exists a constant C such that $C=\max\limits_{x\in E_{pm}}F(n,x_{n+1},x_n)$. Furthermore,

$$\max F(t,u,v)=\max_{(t,u,v)\in R\times X^T}=C>0$$

and $\sigma>0$. We obtain the following inequality.

$$J(x)=x^{\mathrm T}Ax-\sigma\sum_{n=1}^{pm}F(n,x_{n+1},x_n)\geqslant$$

$$\lambda_{\min}\|x\|^2-\sigma\sum_{n=1}^{pm}F(n,x_{n+1},x_n)\geqslant$$

$$\lambda_{\min}\|x\|^2-\sigma Cpm.$$

It is derived that there exist a constant M, such that for every $x \in E_{pm}, J(x) \geqslant M$, That is to say, J has a lower bound.

Furthermore, we choose a sequence $x^{(k)} \in E_{pm}$ that satisfies: for all $k \in N, J(x^{(k)})$ has an upper bound. Then we obtain that there exists a constant M_1, the following inequality holds.

$$\lambda_{\min} \| x^{(k)} \|^2 - \sigma Cpm \leqslant J(x^{(k)}) \leqslant M_1.$$

that is,

$$\lambda_{\min} \| x^{(k)} \|^2 \leqslant M_1 + \sigma Cpm,$$

Then, for any $k \in N$, $\| x^{(k)} \| \leqslant M_2$, where $M_2 = \sqrt{\frac{1}{\lambda_{min}}(M_1 + \sigma Cpm)} \geqslant 0$. Since E_{pm} is finite dimensional, we obtain that there exists a subsequence of $\{x^{(k)}\}$, which is convergent in E_{pm}. J satisfies P-S condition.

Proof of Theorem 3.1 From the above, it has been shown for any $x \in E_{pm}$.

$$J(x) \geqslant \lambda_{\min} \| x \|^2 - \sigma Cpm.$$

Then

$$\inf_{x \in E_{pm}} J(x) > -\infty, \text{and } i_2(J) = 0.$$

On the other hand, by the condition (C_3), we have

$$\lim_{|\rho| \to 0} \frac{F(t,u,v)}{|\rho|^2} = 1, |\rho| = \sqrt{u^2 + v^2}.$$

Choose $0 < \epsilon < \sigma - \frac{9}{4}$, there exists $\delta > 0$, such that

$$\sigma F(t,u,v) \geqslant (\sigma - \epsilon)(u^2 + v^2),$$

where $(u,v) \in B_\delta = \{(u,v) \mid u^2 + v^2 < \delta\}$. Thus, when we choose $\rho = \delta$, we can get

$$J(x) = \sum_{n=1}^{pm} \left[\frac{1}{2} | x_{n+1} - x_n |^2 + \frac{1}{2} | x_n |^2 - \sigma F(n, x_{n+1}, x_n) \right] \leqslant \sum_{n=1}^{pm} \left[2\max\{| x_{n+1} |^2, | x_n |^2\} + \frac{1}{2} | x_n |^2 \right] - \sum_{n=1}^{pm} (\sigma - \epsilon)(| x_{n+1} |^2 + | x_n |^2) \leqslant \sum_{n=1}^{pm} [2(| x_{n+1} |^2 + | x_n |^2)] + \frac{1}{2} | x_n |^2 - 2(\sigma - \epsilon) \times \sum_{n=1}^{pm} | x_n |^2 \leqslant \sum_{n=1}^{pm} \left(4 | x_n |^2 + \frac{1}{2} | x_n |^2 \right) - 2(\sigma - \epsilon) \| x \|^2 = \frac{9}{2} \| x \|^2 - 2(\sigma - \epsilon) \| x \|^2 = 2\left(\frac{9}{4} - \sigma + \epsilon \right) \| x \|^2, \forall x \in E_{pm} \cap B_\delta = B_\delta.$$

Thus, we have

$$\sup_{x \in E_{pm} \cap B_\delta} J(x) < 0, \text{that is}, i_1(J) = pm.$$

By condition (C_1), we obtain that $J(\theta) = 0$. Thus, the proof of theorem 3.1 is complete.

Finally, we present an example to illustrate theorem 3.1.

Example 3.1 Let

$$f(t,u,v,w) = 4v - 4v\left[\left(1 + \sin^2 \frac{\pi t}{m}\right)(u^2 + v^2) + \left(1 + \sin^2 \frac{\pi(t-1)}{m}\right)(v^2 + w^2) \right].$$

Take

$$F(t,u,v) = u^2 + v^2 - \left(1 + \sin^2 \frac{\pi t}{m}\right)(u^2 + v^2)^2,$$

we have

$$F_v'(t,u,v) + F_u'(t-1,v,w) = 4v - 4v \times \left[\left(1 + \sin^2 \frac{\pi t}{m}\right)(u^2 + v^2) + \left(1 + \sin^2 \frac{\pi(t-1)}{m}\right)(v^2 + w^2) \right].$$

This is just the functional $f(t,u,v,w)$. Also, it is easily derived that $F(t,-u,-v) = F(t,u,v)$, $f(t,-u,-v,-w) = -f(t,u,v,w)$, and

$$\lim_{|\rho| \to 0} \frac{F(t,u,v)}{|\rho|^2} = \lim_{|\rho| \to 0} \frac{u^2 + v^2 - \left(1 + \sin^2 \frac{\pi t}{m}\right)(u^2 + v^2)^2}{u^2 + v^2} = 1.$$

In addition, it is noticed that when $u^2 + v^2 > \frac{1}{1 + \sin^2(\pi t/m)}$, $F(t,u,v) < 0$, where the upper bound of $\frac{1}{1 + \sin^2(\pi t/m)}$ for $t \in R$ is 1. Then we have whenever $u^2 + v^2 > 1$, $F(t,u,v) < 0$ for any $t \in R$. To this end, we have tested all the conditions of theorem 3.1 for example 3.1.

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Existence of Multiple Periodic Solutions for a Class of Second-order Nonlinear Difference Equations

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[**Abstract**] The Z_2-group index theory is used to explored the periodic solutions for a class of second-order nonlinear difference equations. The sufficient conditions responsible for the existence of multiple periodic solutions of this kind of discrete system are obtained. The proofs are presented in detail. Furthermore, an example is given to illustrate the conclusions.

[**Key words**] periodic solution　　nonlinear difference equations　　Z_2-group index theory